Finite Math - Spring 2017

Lecture Notes - 2/22/2017

Homework

- Section 3.3 - 7, 9, 11, 15, 17, 19, 27, 31, 33, 39

Section 3.3 - Future Value of an Annuity; Sinking Funds

Annuities. At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account? An *annuity* is a sequence of equal periodic payments. If payments are made at the end of each time interval, then the annuity if called an *ordinary annuity*. Our goal in this section will be to find the future value of an annuity.

Example 1. *Suppose you decide to deposit* \$100 *every* 6 *months into a savings account which pays* 6% *compounded semiannually. If you make* 6 *deposits, one at the end of each interest payment period over the course of* 3 *years, how much money will be in the account after the last deposit is made?*

Solution. *We can visualize the value of each of those* \$100 *deposits in a table.*

Deposit	Term Deposited	Number of times Compounded	Future Value
\$100	1	5	$\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$
\$100	2	4	$\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$
\$100	3	3	$\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$
\$100	4	2	$\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$
\$100	5	1	$\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$
\$100	6	0	$\$100\left(1+\frac{0.06}{2}\right)^0 = \$100$

So adding up the future values of all these will give us the amount of money in the account

$$B = \$100(1.03)^5+\$100(1.03)^4+\$100(1.03)^3+\$100(1.03)^2+\$100(1.03)+\$100 = \$646.84$$

There is actually a nice trick to adding this up. Begin by writing the same list, but let's multiply it by 1.03

$$1.03B = \$100(1.03)^6+\$100(1.03)^5+\$100(1.03)^4+\$100(1.03)^3+\$100(1.03)^2+\$100(1.03)$$

then we compute $1.03B - B$:

$$\begin{array}{rcccccccccccccc} 1.03B & = & \$100(1.03)^6 & + & \cancel{\$100(1.03)^5} & + & \cancel{\$100(1.03)^4} & + & \cancel{\$100(1.03)^3} & + & \cancel{\$100(1.03)^2} & + & \cancel{\$100(1.03)} & & \\ -B & = & & - & \cancel{\$100(1.03)^5} & - & \cancel{\$100(1.03)^4} & - & \cancel{\$100(1.03)^3} & - & \cancel{\$100(1.03)^2} & - & \cancel{\$100(1.03)} & - & \$100 \end{array}$$

So we arrive at

$$1.03B - B = .03B = \$100(1.03)^6 - \$100 = \$100\left((1+0.03)^6 - 1\right)$$

Solving for B gives

$$B = \$100\frac{(1+0.03)^6 - 1}{0.03} = \$646.84.$$

This gives rise to the following formula

Definition 1 (Future Value of an Ordinary Annuity)**.**

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

where

$$\begin{aligned} FV &= \textit{future value} \\ PMT &= \textit{periodic payment} \\ i &= \textit{rate per period} \\ n &= \textit{number of payments (periods)} \end{aligned}$$

Note that the payments are made at the end of each period.

In the above formula, $i = \dfrac{r}{m}$, where r is the interest rate (as a decimal) and m is the number compounding periods per year and $n = mt$ where t is the length of time of the investment. We can rewrite the formula with r and m instead of i

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

Example 2. *What is the value of an annuity at the end of* 10 *years if* \$1,000 *is deposited every* 3 *months into an account earning* 8% *compounded quarterly. How much of this value is interest?*

Solution. *In this problem,* $PMT = \$1,000$*,* $r = 0.08$*,* $m = 4$*, and* $n = 4(10) = 40$*. So* $\dfrac{r}{m} = \dfrac{0.08}{4} = 0.02$ *and the future value is*

$$FV = \$1,000\frac{\left(1+\frac{0.08}{4}\right)^{40} - 1}{0.08/4} = \$1,000\frac{(1.02)^{40} - 1}{0.02} = \$60,401.98$$

To figure out how much is interest, we simply figure out how much money we put into the account, then subtract that from the future value. Since we made 40 *payments of* \$1,000*, we invested* $40(\$1,000) = \$40,000$*. Thus the interest is*

$$I = \$60,401.98 - \$40,000 = \$20,401.98.$$

Example 3. *If* \$1,000 *is deposited at the end of each year for* 5 *years into an ordinary annuity earning* 8.32% *compounded annually, what will be the value of the annuity at the end of the* 5 *years?*

Solution. \$5,904.15

Sinking Funds. We can turn the annuities picture around and ask how much we would need to deposit into an account each period in order to get the desired final value. It is simple to solve for PMT in the annuities formula to get

Definition 2 (Sinking Funds)**.**

$$PMT = FV\frac{i}{(1+i)^n - 1}$$

where all the variables have the same meaning as for annuities.

Example 4. *Let's revisit those new parents who are trying to save for their child's college and examine the more likely case that they will make payments into a savings account. They still want to save up* $80,000 *in* 17 *years and they have found an account that will pay* 8% *compounded quarterly. How much will they have to deposit every quarter in order to have a value of* $80,000*?*

Solution. *In this case,* $i = \frac{r}{m} = \frac{0.08}{4} = 0.02$*,* $n = 4(17) = 68$ *and* $FV = \$80,000$*, so the required payment is*

$$PMT = \$80,000\frac{0.02}{(1.02)^{68} - 1} = \$562.54.$$

Thus the parents would have to make a deposit of $562.54 *every* 3 *months in order to have the desired* $80,000 *after* 17 *years.*

Example 5. *A bond issue is approved for building a marina in a city. The city is required to make regular payments every* 3 *months into a sinking fund paying* 5.4% *compounded quarterly. At the end of* 10 *years, the bond obligation will be retired with a cost of* $5,000,000*. How much will the city have to pay each quarter?*

Solution. $95,094.67